Latent Variables and Lossless Compression

James Townsend

Stanford IT Forum
28/10/2022

Talk outline

1. Latent variable methods
2. Lossless compression and asymmetric numeral systems
3. Combining 1 and 2

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Suppose modelling x is *difficult* in some way, it might be

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This is called a *latent* variable

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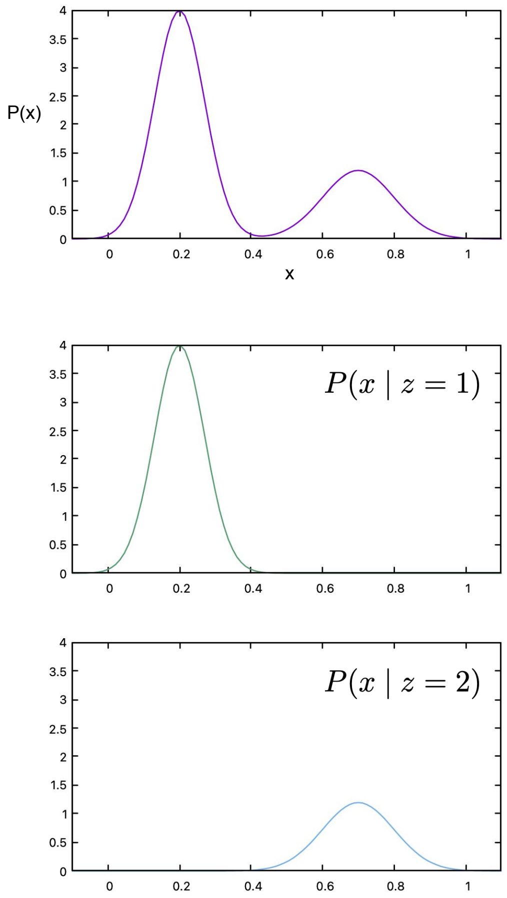
‘Latent variable methods’

$$P(x) = c_1 N(x; \mu_1, \sigma_1) + c_2 N(x; \mu_2, \sigma_2)$$



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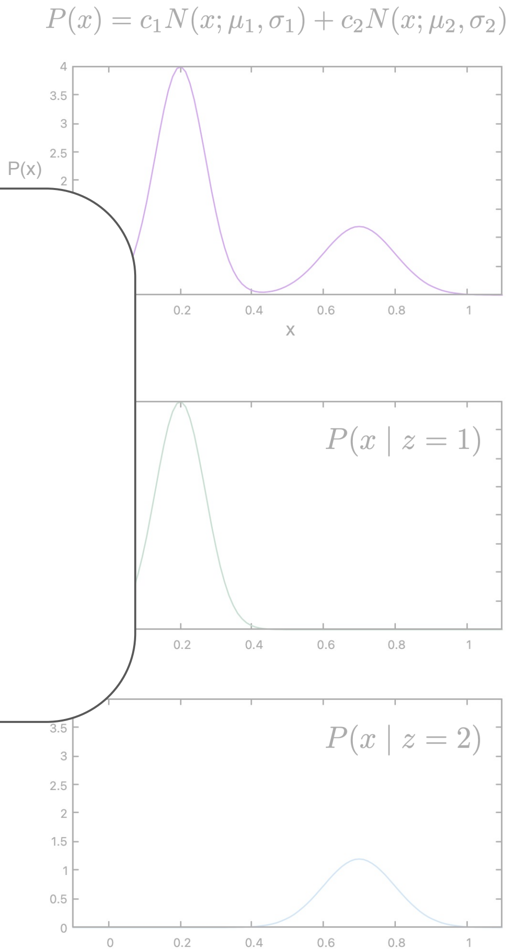
Claim: this is a common pattern.
It's useful to notice it.



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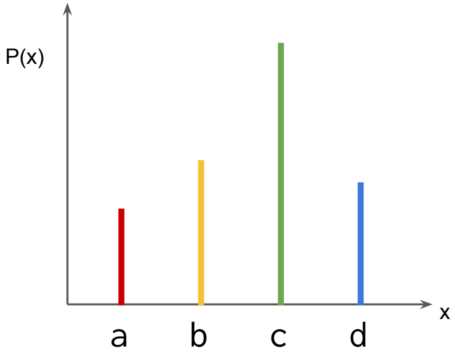

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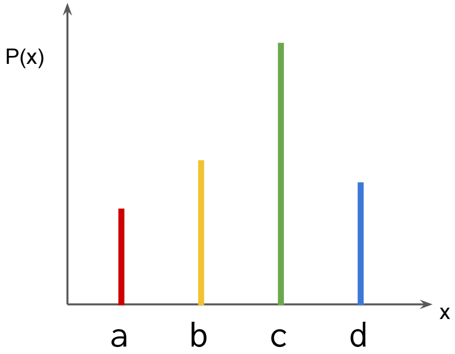

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Answer: compute

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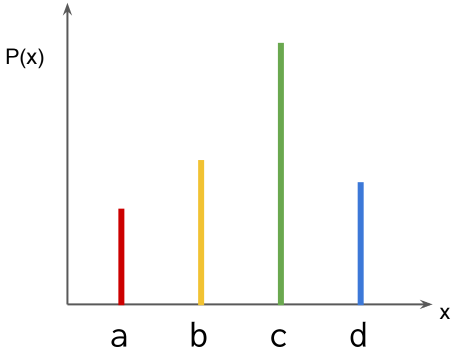

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In `numpy.random.choice()`:

```
cdf = p.cumsum()
cdf /= cdf[-1]
uniform_samples = self.random_sample(shape)
idx = cdf.searchsorted(uniform_samples, side='right')
# searchsorted returns a scalar
# force cast to int for LLP64
idx = np.array(idx, copy=False).astype(int, casting='unsafe')
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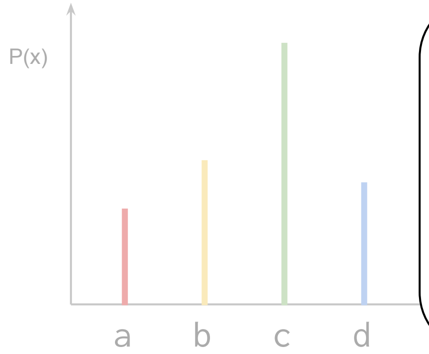

The variable u is a *latent variable*.

We have P(x) = ∫P(x, u) du
and H(x, u) > H(x).

b	c	d
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```
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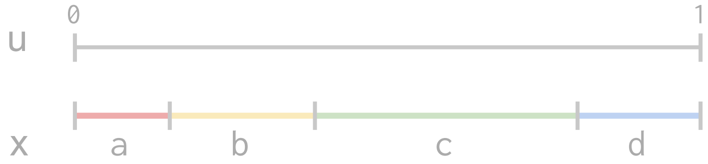
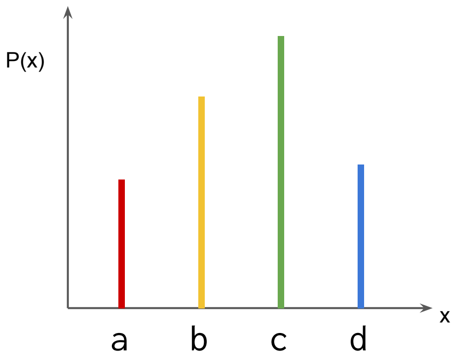


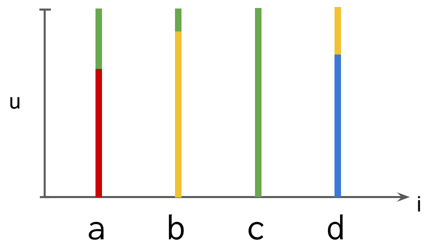
Example 2: the alias method (Walker, 1974)

Fast sampling from a categorical...

...2 table lookups + 2 samples from uniform distribution, *no search*.



x	a	b	c	d
P(x)	0.17	0.28	0.36	0.19



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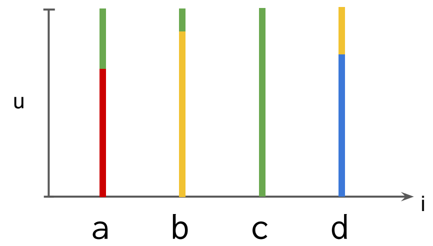
```
sample i ~ Uniform { a , b , c , d }
sample u ~ Uniform [0, 1)

if u < P(switch | i):
  x = alias(x)
else:
  x = i
```



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What's the latent variable this time?

It's the pair (u, i):

$$P(x) = \int P(x, u, i)\, du\, di$$
$$\text{and } H(x, u, i) > H(x)$$

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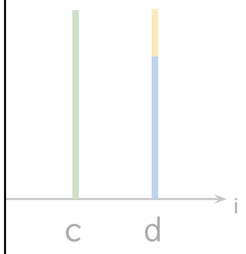

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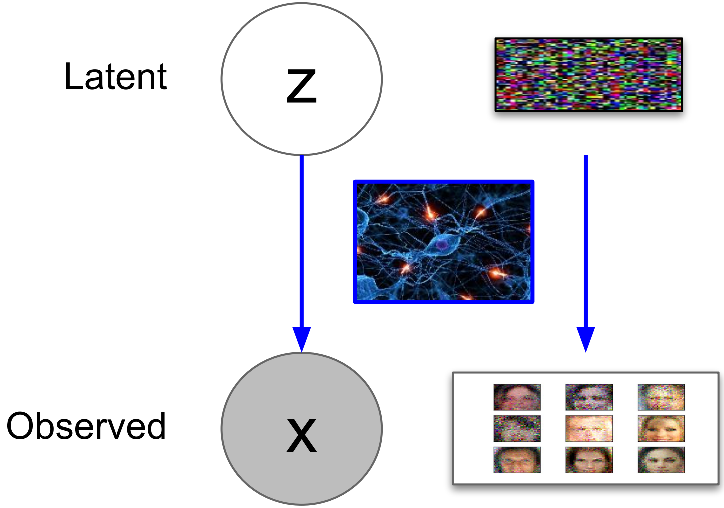
Example 3: latent variable *models* (VAEs)



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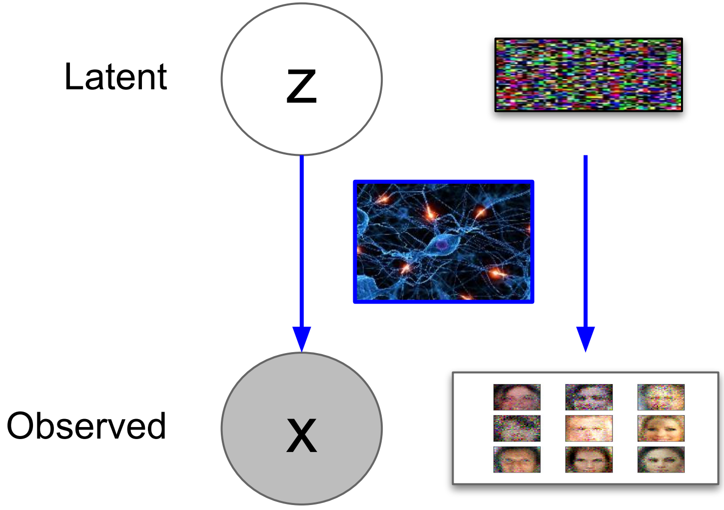

Joint distribution p(x, z) = p(x | z)p(z) factors:

Prior over latent z

$$p(z) = N(z; 0, I)$$

Conditional over x

$$p(x \mid z; \theta) = m(x; \mu(z; \theta))$$

- μ is a **neural network,** θ parameters

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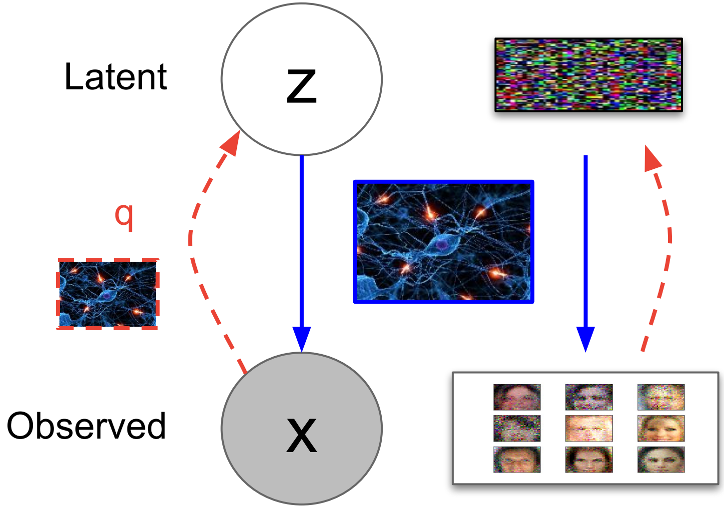

Optimize ***lower bound*** on marginal

$$\log p(x) \geq L(\theta, \varphi) \triangleq E_{q(z;\, x)}[\log p_\theta(x, z) - \log q_\varphi(z; x)]$$

$$q_\varphi(z; x) = N(z; \mu_q(x; \varphi), \Sigma_q(x; \varphi))$$

- q **approximates the posterior**

$$q(z; x) \approx p(z \mid x)$$

- $\Sigma_q(x; \varphi)$ usually **diagonal** (‘mean field’)

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Compressing uniform data

A relatively simple ‘arithmetic coding’ method:

```
# Assuming s in {0, 1, ...} and u in {0, 1, ..., N - 1}
encode(s, u) := N * s + u

# Reverse operation:
decode(s’) := (s’ // N, s’ % N)
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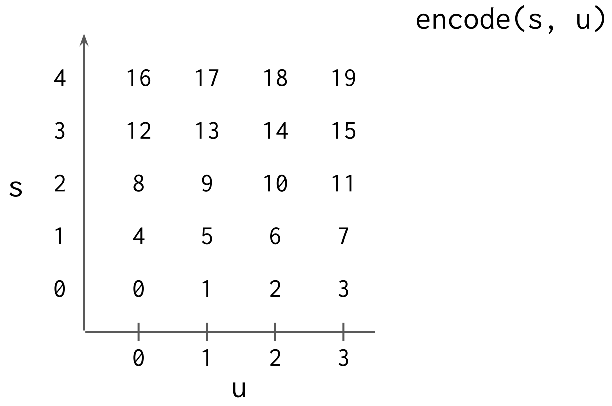

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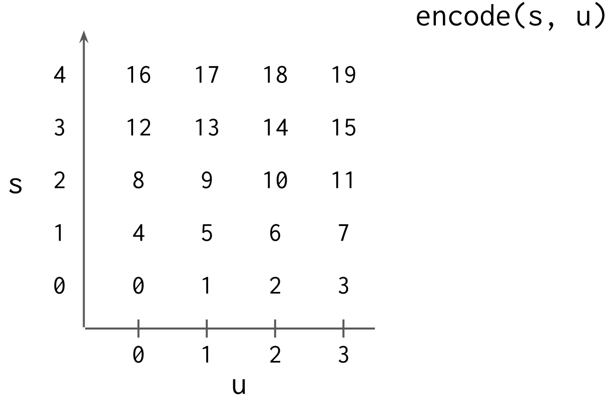

Can compress/decompress a list:

```
s = 0
for u in reversed(us):
  s = encode(s, u)

# to undo the above:
us = []
for i in range(length):
  s, u = decode(s)
  us.append(u)
```

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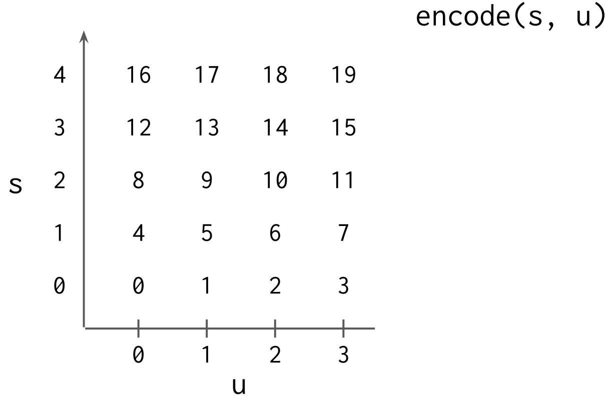

Genius work by Duda (2009) showed that

- You can address the issue of `s` growing
- `size(encode(s, u)) ≈ size(s) + log2(N)`. Great if u is really uniform distributed, because then H(u) = log2(N).

Intuitively: `log2(encode(s, u)) = log2(N * s + u) ≈ log2(s) + log2(N)`

Jarek Duda

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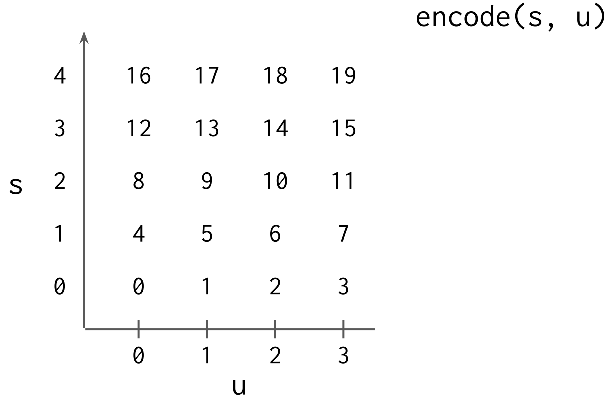

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This guy has done two PhDs!

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If we have

- an encoder/decoder for (X, Z), $\Delta\text{size}(s) = H(X, Z)$
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```
def encode_x(s, x):
  s, z = decode_zgx(x)(s)       # - log (1 / P(z | x))
  s    = encode_xz(s, (x, z))  # + log (1 / P(z, x))
  return s

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  s         = encode_zgx(x)(s, z)
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= log 1 / P(x)

‘BB-ANS’
Townsend et al. (2019)

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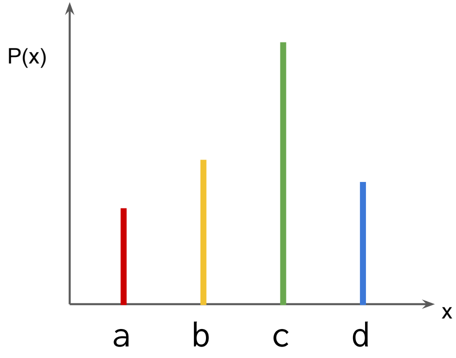

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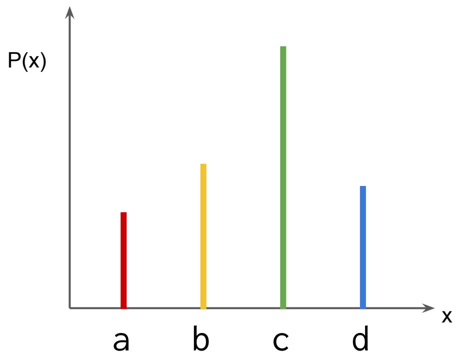

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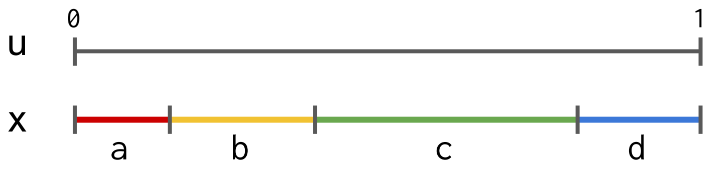

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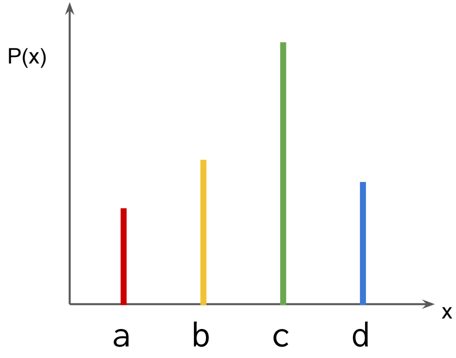

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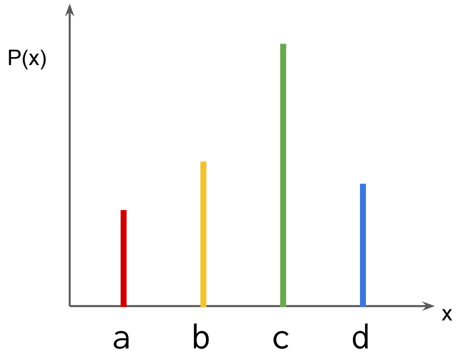
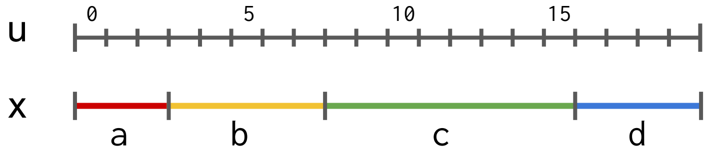


x	a	b	c	d
P(x)	3/20	5/20	8/20	4/20



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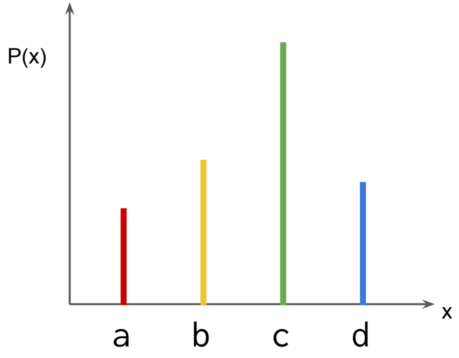

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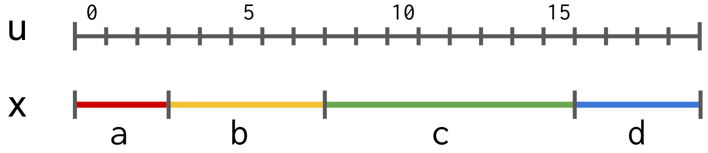
M(x)	3	5	8	4



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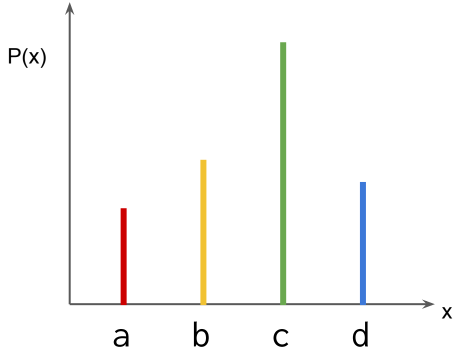

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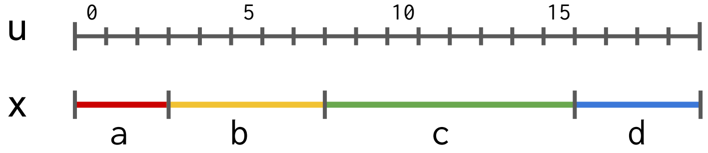

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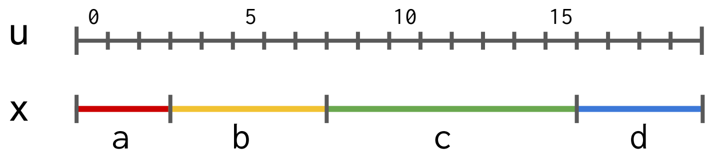

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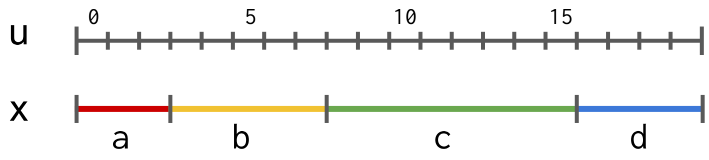

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def decode_x(s):
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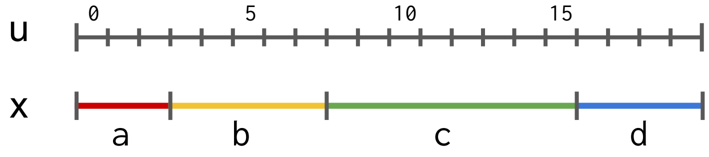

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```
def encode_xu(s, (x , u)):
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  # u...
  return encode_uniform(20)(s, u)
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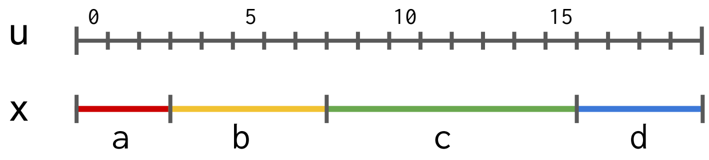

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  return encode_uniform(20)(s, u)

def decode_xu(s):
  # First decode u
  s, u = decode_uniform(20)(s)
  # Then search
  x = cdf_lookup(u)
  return s, (x, u)
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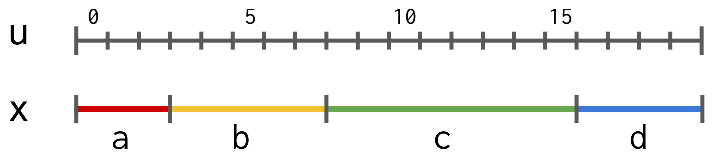

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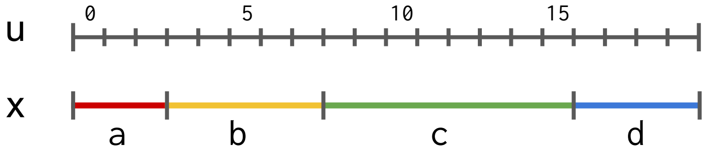

x	a	b	c	d
M(x)	3	5	8	4

```
def encode_x(s, x):
  s, u = decode_ugx(x)(s)
  s    = encode_xu(s, (x, u))
  return s

def decode_x(s):
  s, (x, u) = decode_xu(s)
  s         = encode_ugx(x)(s, u)
  return s
```

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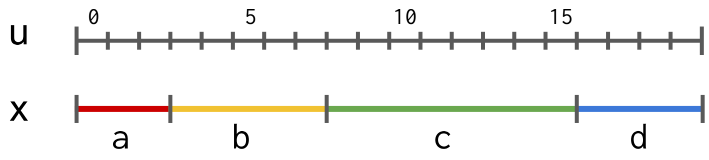

x	a	b	c	d
M(x)	3	5	8	4
F(x)	0	3	8	16

```
def encode_x(s, x):
  s, u = decode_ugx(x)(s)
  s    = encode_xu(s, (x, u))
  return s

def decode_x(s):
  s, (x, u) = decode_xu(s)
  s         = encode_ugx(x)(s, u)
  return s
```

```
def encode_ugx(x):
  def enc(s, u):
    # u ~ Uniform {F(x), F(x)+1, ..., F(x) + M(x)}
    return encode_uniform(M(x))(u - F(x))
  return enc

def decode_ugx(x):
  def dec(s):
    s, i = decode_uniform(M(x))(s)
    return s, i + F(x)
  return dec
```

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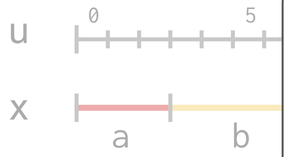
How do you compress a discrete random variable?



c	d
8	4
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This is the range variant of asymmetric numeral systems (rANS)!

```
def encode_x(s, x):
  s, u = decode_ugx(
  s    = encode_xu(s
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def decode_x(s):
  s, (x, u) = decode_xu(s)
  s         = encode_ugx(x)(s, u)
  return s
```

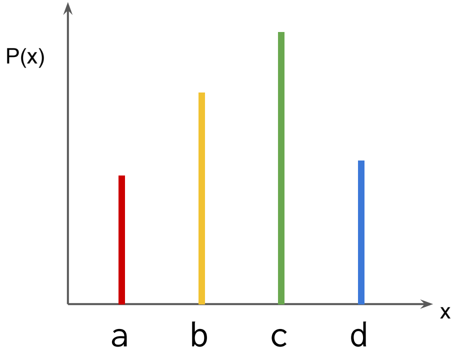
```
, F(x) + M(x)}
F(x))
```

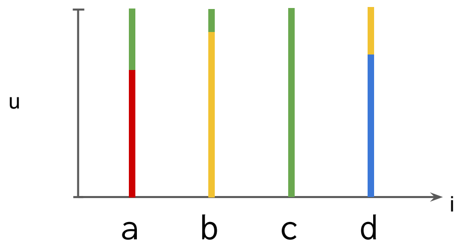
```
  def dec(s):
    s, i = decode_uniform(M(x))(s)
    return s, i + F(x)
  return dec
```

Example 2: the alias method (Walker, 1974)

Fast sampling from a categorical...

...2 table lookups + 2 samples from uniform distribution, *no search*.





x	a	b	c	d
P(x)	0.17	0.28	0.36	0.19

x	a	b	c	d
alias(x)	c	c	-	b
P(switch | i)	0.32	0.12	0.00	0.24

Example 2: the alias method ~~(Walker, 1974)~~

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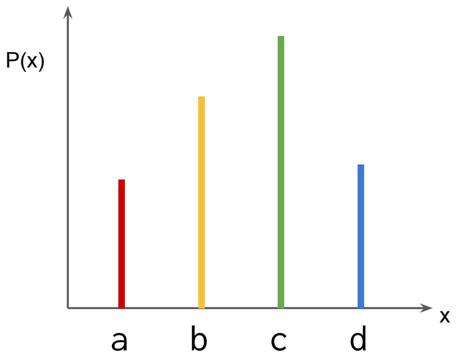

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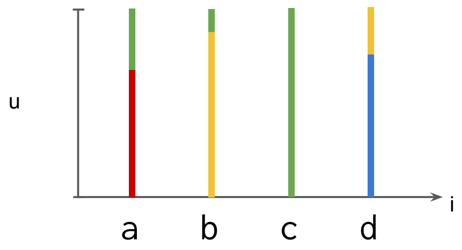

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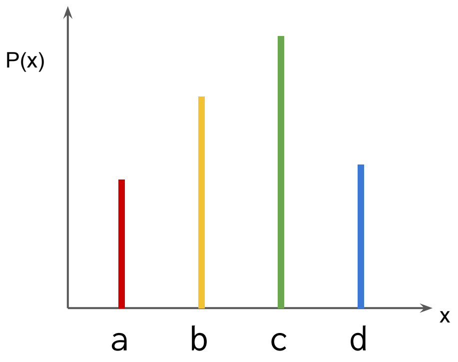
Fast decoding from a categorical...

...2 table lookups + 2 decodes from uniform distribution, *no search*.



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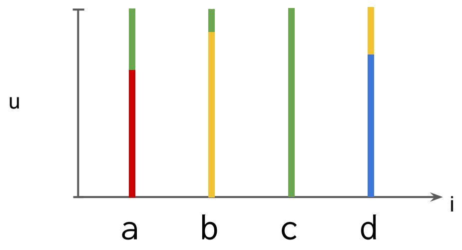

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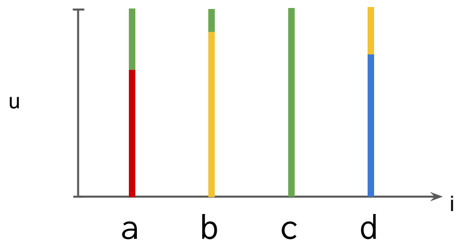

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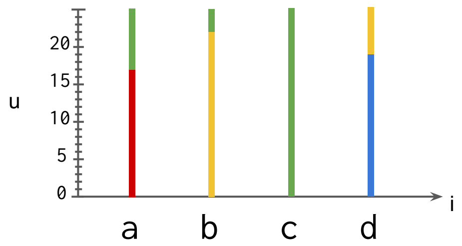

x	a	b	c	d
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M(switch | i)	8	3	0	6

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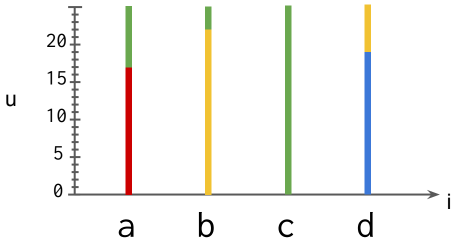
```
def encode_x(s, x):
  s, (u, i) = decode_uigx(x)(s)
  s         = encode_xui(s, (x, u, i))
  return s

def decode_x(s):
  s, (x, u, i) = decode_xui(s)
  s            = encode_uigx(x)(s, (u, i))
  return s
```



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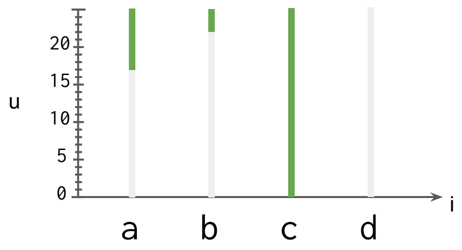
(u, i) | x = c



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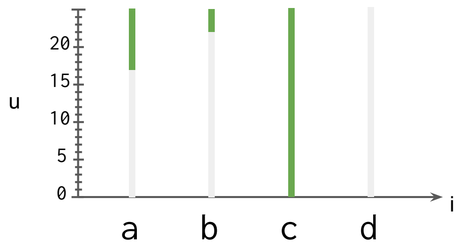


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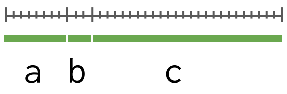

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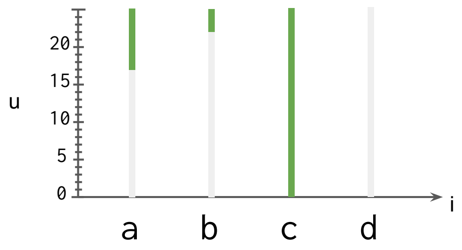

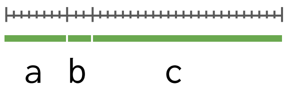

Decoding (u, i) | x requires search!

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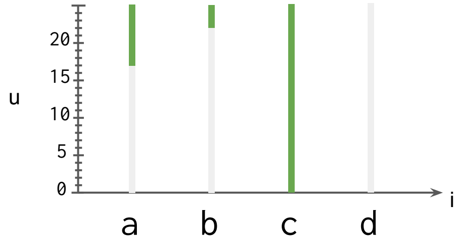


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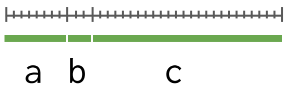

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*during decoding.

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```
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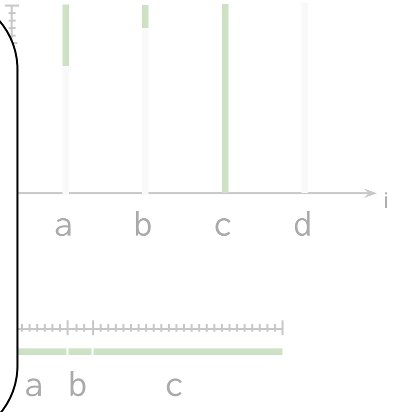
  return s

def decode_x(s):
  s, (x, u, i) = decode_xui(
  s            = encode_uigx
  return s
```

Key point: alias method *moves* work from the decoder to the encoder...

total complexity of encoding + decoding stays the same
🤔

(u, i) | x = c



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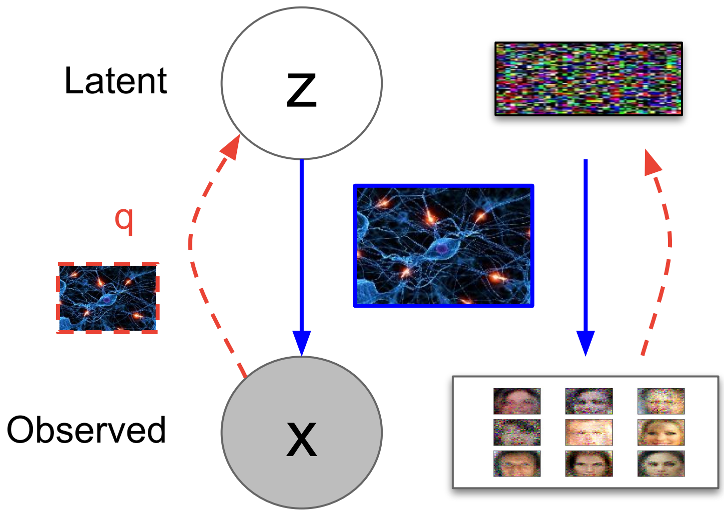
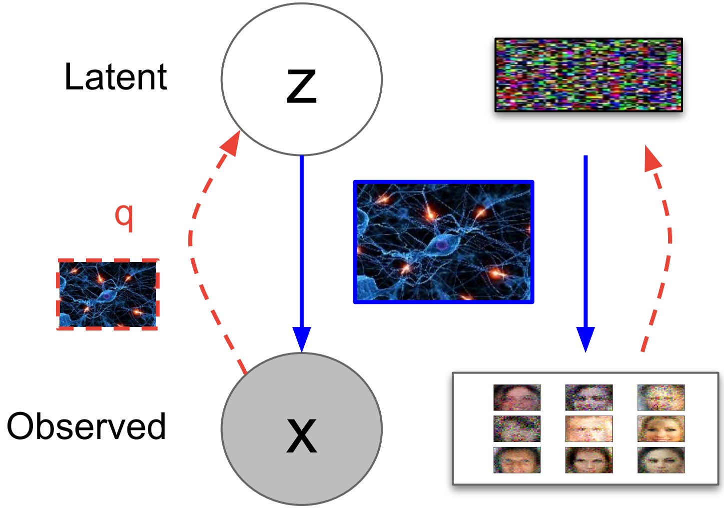
Example 3: latent variable *models*



Example 3: latent variable *models*

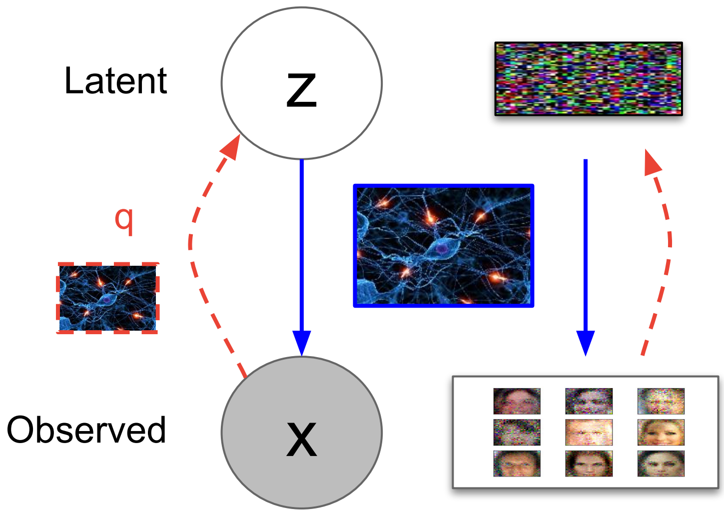


Train a VAE model on images...

...directly apply latent variable compression.

Result: a good (formerly SOTA) lossless image compression rate, with fast-ish decoding.

Example 3: latent variable *models*



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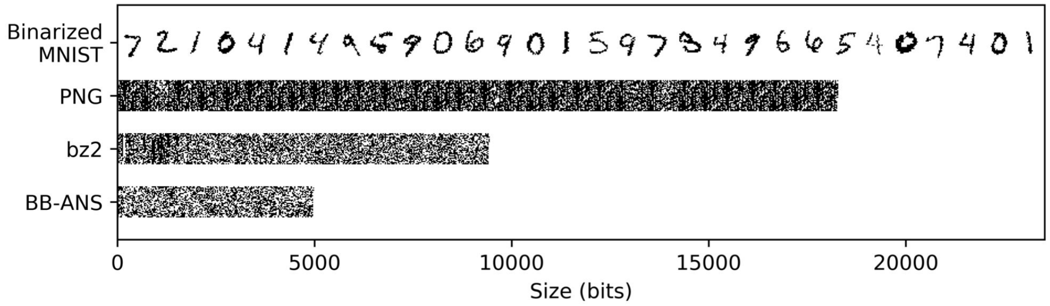

Townsend et al. (2019)

More!

- Optimal compression of *multisets* (Severo et al., 2022)
- Optimal compression of *unlabelled random graphs* (unpublished)
- Compression with *latent state space models* (Townsend and Murray, 2021)
- Compression with *hierarchical LVMs* (Townsend et al., 2020; F. Kingma et al. 2020)
- Compression with *diffusion models* (D. Kingma et al., 2021)

More!

- Optimal compression of *multisets* (Severo et al., 2022)
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- Compression with *diffusion models* (D. Kingma et al., 2021)

All use the pattern discussed in this talk. There are probably more examples, still to be discovered.

The end. Thanks for listening.

James Townsend

Stanford IT Forum
28/10/2022

References

1. Walker, A. J. (April 1974). "New fast method for generating discrete random numbers with arbitrary frequency distributions". *Electronics Letters*. **10** (8): 127.
2. Duda, J. (2009). “Asymmetric numeral systems”. https://arxiv.org/abs/0902.0271.
3. Townsend, James, Thomas Bird, and David Barber. ‘Practical Lossless Compression with Latent Variables Using Bits Back Coding’, 2019. https://openreview.net/forum?id=ryE98iR5tm.
4. Daniel Severo*, James Townsend*, Ashish Khisti, Alireza Makhzani, and Karen Ullrich, Compressing Multisets with Large Alphabets, appearing at the Data Compression Conference (DCC), 2022.
5. James Townsend and Iain Murray, Lossless Compression with State Space Models Using Bits Back Coding, Neural Compression: From Information Theory to Applications -- Workshop @ ICLR 2021.
6. James Townsend*, Thomas Bird*, Julius Kunze, and David Barber, HiLLoC: Lossless Image Compression with Hierarchical Latent Variable Models, International Conference on Learning Representations (ICLR), 2020. *Equal contribution.
7. Friso Kingma, Pieter Abbeel, Jonathan Ho. Bit-Swap: Recursive Bits-Back Coding for Lossless Compression with Hierarchical Latent Variables. *Proceedings of the 36th International Conference on Machine Learning*, PMLR 97:3408-3417, 2019.
8. Kingma, Diederik P., Tim Salimans, Ben Poole, and Jonathan Ho. ‘Variational Diffusion Models’, 2022. https://openreview.net/forum?id=2LdBqxc1Yv.